SOME PROBLEMS IN APPROXIMATION THEORY

by

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TABLE OF CONTENTS

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A classical theorem of K. Weierstrass states if f is a real valued, continuous, function defined on a compact interval, then there is a sequence of polynomials (P_n) which converges uniformly to f. In this work four such sequences are examined.

In each case P_n is determined by finitely many values of f, and the main problem is to find an estimate of $||P_n - f||$.

The first sequence considered is the sequence of the most economical stable interpolatory polynomials introduced by E. Egerváry and P. Turán. The next two sequences are formed by applying summability methods to the sequence of Lagrange interpolation polynomials on suitable nodes. One of these sequences gives a new proof of the famous theorem of A. F. Timan. Finally, a slight change in the sequence of polynomials which were used to prove Timan's theorem leads to a proof of a theorem of S. A. Teljakovskii.

CHAPTER I

INTRODUCTION

1. Introduction

Let $C([-1,1])$ with the supremum norm be the space of real valued, continuous functions defined on the interval $[-1,1]$. Likewise we shall denote the space of real valued, continuous, 2π-periodic functions with the supremum norm by $C([0,2\pi])$. We shall use the symbol $C(X)$ to denote either space. In this work we shall be concerned with uniform approximation of functions in these spaces by polynomials. Naturally in the first space we shall be considering algebraic polynomials of the form $\sum_{i=0}^{n} a_i x^i$ and in the second space we shall consider trigonometric polynomials such as $\sum_{i=0}^{n} (a_i \cos i\, x + b_i \sin i\, x)$.

In 1885, K. Weierstrass [30] proved that each function in $C(X)$ is the limit of a uniformly convergent sequence of polynomials. This theorem founded modern approximation theory. Since then two particularly important types of problems have been intensely studied. On one hand the constructive theory of functions has created many new types of questions and results. On the other hand, the investigations of particular methods of approximation have

also been very rewarding. I shall elaborate on these problems because they form an important background for the chapters to follow.

2. On the Constructive Theory of Functions

The first step in building a constructive theory of functions was made by D.Jackson[13].Let $f \in C([-1,1])$. Suppose that we define the best approximation of f by polynomials of degree n by

$$E_n(f) = \inf_{P_n} ||f - P_n||$$

where P_n ranges over all algebraic polynomials of degree n. Similarly, for $f \in C([0,2\pi])$ we shall consider the quanitity

$$E_n^*(f) = \inf_{T_n} ||f - T_n||$$

where T_n ranges over all trigonometric polynomials of degree n. Jackson considered the problem of estimating $E_n(f)$ and $E_n^*(f)$. To describe his results we need the following definition.

Definition. If $f \in C(X)$, then the modulus of continuity of f is a function ω such that

$$\omega(h) = \sup_{|x-y|\le h} |f(x) - f(y)|.$$

Now Jackson's theorems may be easily stated.

Theorem 1. (Jackson) There is a constant A such that

$$E_n^*(f) \le A\,\omega(\tfrac{1}{n}) , \quad n = 1,2,3,...$$

for all $f \in C([0,2\pi])$.

Theorem 2. (Jackson) There is a constant B such that

$$E_n(f) \leq B\,\omega(\tfrac{1}{n}), n = 1,2,3,\ldots$$

for all $f \in C([-1,1])$.

(The letters A,B,C, will be used to denote absolute constants. Repeated use of the same letter should not be interpreted as signifying the same constant recurring.)

Two important corollaries of Theorems 1 and 2 deserve mentioning. Let $MH_\alpha(X)$ be the class of functions f in $C(X)$ such that $|f(x) - f(y)| \leq M|x-y|^\alpha$ for all x and y in X. Then it is not hard to see that $f \in MH_\alpha(X)$ if and only if $\omega(h) \leq Mh^\alpha$ for all $h \geq 0$. Therefore the following are direct consequences of Jackson's theorems.

Corollary 1. Let $0 < \alpha \leq 1$. If $f \in MH_\alpha([0,2\pi])$, for some constant M, then

$$E_n^*(f) \leq \frac{A}{n^\alpha}, \; n = 1,2,\ldots$$

for some constant A.

Corollary 2. Let $0 < \alpha \leq 1$. If $f \in MH_\alpha([-1,1])$ for some constant M, then

$$E_n(f) \leq \frac{B}{n^\alpha}, \; n = 1,2,\ldots$$

for some constant B.

In these corollaries, Jackson showed that a well known class of functions has certain approximation properties. Then S. N. Bernstein proved the following converse result.

Theorem 3. (Bernstein, [3]) If $f \in C([0,2\pi])$ and $E_n^*(f) \leq \frac{A}{n^\alpha}$ for some α satisfying $0 < \alpha < 1$, a fixed constant A, and all natural numbers n, then there is a constant M such that $f \in MH_\alpha([0,2\pi])$.

The remaining case of $\alpha = 1$ remained an open problem for many years until A. Zygmund [31] settled it. He considered the class of functions $f \in C([0,2\pi])$ such that

$$|f(x+h) - 2f(x) + f(x-h)| \leq M|h|$$

for all possible x and h. We shall denote this class by MZ. His principal contribution to the constructive theory of functions was the following theorem.

Theorem 4. (Zygmund) Let $f \in C([0,2\pi])$. Then $E_n^*(f) \leq \frac{A}{n}$ for $n = 1,2,\ldots$ and some absolute constant A if and only if $f \in MZ$ for some constant M.

The reader may have noticed the sudden disappearance of the space $C([-1,1])$ from the discussion. This is because there are no analogous converse theorems in this case. In order to have a matching direct and converse theorem A. F. Timan [26] noticed that the following strengthening of Jackson's Theorem 2 was necessary.

Theorem 5. (Timan) There is a constant A such that if $f \in C([-1,1])$ and n is a natural number, then there is a polynomial P_n of degree n such that

$$|f(x) - P_n(x)| \leq A\left(\omega\left(\frac{\sqrt{(1-x^2)}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right)$$

for all x in the interval $[-1,1]$.

Now one can see that the following theorem is the type of "constructive" theorem which is to be expected in $C([-1,1])$.

Theorem 6. (V. K. Dzjadyk [5].) Let $f \in C([-1,1])$. Suppose

that $0 < \alpha < 1$. Then, there is a constant A such that, to each n there corresponds a polynomial P_n of degree n such that

$$|f(x) - P_n(x)| \le A\left(\frac{\sqrt{(1-x^2)}}{n} + \frac{1}{n^2}\right)^{\alpha}$$

if and only if

$$\omega(h) \le Bh^{\alpha}$$

for some constant B.

This concludes the brief discussion of the constructive theory of functions: a much more detailed presentation can be found in the treatise by A. F. Timan [27].

3. An Interesting Method of Approximation

Of particular methods of approximation one important example is very well known: Hermite - Fejér Interpolation. Consider the triangular array of numbers

$$\begin{array}{cccc} x_1^{(1)} & & & \\ x_1^{(2)} & x_2^{(2)} & & \\ x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

where

$$-1 \le x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} \le 1.$$

If $f \in C([-1,1])$ then we associate with $f(x)$ that unique polynomial $H_n[f,x]$ of degree $2n - 1$ such that

$$H_n[f,x_i^{(n)}] = f(x_i^{(n)}), \quad i = 1,2,\ldots,n$$

$$H_n'[f,x_i^{(n)}] = 0 \quad, \quad i = 1,2,\ldots,n$$

A classical result of L. Fejér [7] states that if, for each n, the nodes $x_1, x_2, \ldots, x_n$ are chosen to be the zeros of $T_n(x) = \cos n(\arccos x)$, then $H_n[f,x]$ converges to $f(x)$ uniformly on the interval $[-1,1]$ as n tends to infinity. This particular convergence process has been extensively studied.

A simply stated, but as yet unsolved, problem is to calculate the exact value of $e(n,f) = ||f - H_n(f)||$ in terms of n and f. The simplicity of this operator H_n has attracted many authors to this problem. Below is a brief outline of some of the major developments in this problem because they are indicative of the types of questions asked about other operators. In particular, these developments were very helpful in considering a similar problem treated in Chapter II.

The first estimate of $e(n,f)$ was made by a Romanian mathematician E. Moldovan [18].

Theorem 7. (Moldovan) There is a constant A such that

$$e(n,f) \leq A\,\omega\left(\frac{\log n}{n}\right)$$

for all f and all n.

This result was later generalized when O. Shisha and B. Mond [23] considered a Hermite-Fejér operator for functions of several variables.

More recently, R. Bojanic [2] has proved the inequality

$$e(n,f) \leq \frac{B}{n} \cdot \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) \leq A\,\omega\left(\frac{\log n}{n}\right)$$

which is an improvement of Moldovan's result.

The next step is to examine the error for particular subclasses of $C([-1,1])$. The following unpublished result has been obtained by J. Szabodos.

Theorem 8. (Szabodos) $||H_n(f) - f|| = O(\frac{1}{n})$ if and only if $g(t)$ and $\tilde{g}(t) \in MH_1([0,\pi])$ for some constant M where $g(t) \equiv f(\cos t)$ and $\tilde{g}$ is the conjugate function of g.

This type of result is very unusual in interpolation theory. It is to be hoped that more results of this type will produce general methods of attack to assist in learning more about particular approximating operators in $C([-1,1])$.

The final stage in the investigation of H_n would be to calculate the best constants. These types of problems are the most difficult and hence the most challenging.

In this work we examine four different approximation methods which arise from interpolation theory. In particular the degree of approximation, or the size of the error, is the most important aspect to be considered here. However, each method has its own intrinsic importance and related unsolved problems. Because of the very technical nature of this dissertation, the rest of this chapter will be devoted to a general discussion of the results and the other aspects just mentioned.

4. Remarks on Chapter II

In Chapter 2 we prove a qualitative version of a theorem of E. Egerváry and P. Turán. Let $x_0, x_1, \ldots, x_{n+1}$ be n+2 distinct points in [-1,+1]. Furthermore suppose

that $r_0(x), r_1(x), \ldots, r_{n+1}(x)$ are polynomials such that

$$r_i(x_j) = 1 \quad \text{if} \quad i = j$$
$$= 0 \quad \text{if} \quad i \neq j$$

for each pair (i,j). Now we may construct an interpolation polynomial

$$R_n[f,x] = \sum_{k=0}^{n+1} f(x_k)r_k(x)$$

for any function f defined on $[-1,1]$.

This interpolation process is said to be stable, if, for each f and each x, we have

$$\min_k f(x_k) \leq R_n[f,x] \leq \max_k f(x_k).$$

Now we introduce the parameter

$$\deg R_n = \sum_{k=0}^{n+1} (\text{degree of } r_k(x)).$$

The most economical interolation process is that stable process for which $\deg R_n$ is minimal.

Egerváry and Turán [6] proved the existence and uniqueness of such a polynomial $R_n[f,x]$ and gave an explicit representation of it. Their paper, by the way, is an excellent example of elegant classical analysis. They went on to show that $R_n[f,x]$ converges uniformly to $f(x)$ on $[-1,1]$ as n tends to infinity, provided f is continuous.

In Chapter 2 it is proved that

$$||R_n(f) - f|| \leq \frac{A}{n} \sum_{k=1}^{n} \omega(\tfrac{1}{k}).$$

Notice the similarity of this result to that obtained by R. Bojanic for Hermite-Fejér interpolation. Because R_n is a positive linear operator, it has been possible to find

a lower bound for the error which shows that the estimates are best possible in some sense.

The theorem presented in this chapter raises a natural question. Let us recall the theorem of D. Jackson which says that $E_n(f) \leq B\,\omega(\frac{1}{n})$ where B is an absolute constant. Therefore we may ask whether it is possible to find a constant C such that, for all n and all continuous f,

$$||R_n(f) - f|| \leq \frac{C}{n} \sum_{k=1}^{n} E_k(f),\ n = 1,2,\ldots .$$

Another interesting question arises from further properties of R_n. Egerváry and Turán showed that the nodes $x_0, x_1, \ldots, x_{n+1}$ corresponding to this optimal process must be the zeros of $(1-x^2)P_n(x)$ where P_n is the Legendre polynomial of degree n. Now it is also true that

$$R_n[f,x_i] = f(x_i) \quad \text{if} \quad i = 0,1,\ldots,n+1$$

and

$$R_n'[f,x_i] = 0 \quad \text{if} \quad i = 1,2,\ldots,n.$$

So let $y_0, y_1, \ldots, y_{n+1}$ be the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ where $P_n^{(\alpha,\beta)}(x)$ is the n th Jacobi polynomial with parameters α and β. It is easy to find the unique polynomial $S_n[f,x]$ of degree 2n+1 such that

$$S_n[f,y_i] = f(y_i) \quad \text{for} \quad i = 0,1,\ldots,n+1$$

and

$$S_n'[f,y_i] = 0 \quad \text{for} \quad i = 1,2,\ldots,n.$$

Naturally if $y_i = x_i$ (i.e. $\alpha = \beta = 0$) then $S_n = R_n$. Furthermore S_n will not necessarily converge to f uniformly in

$[-1,1]$. It would be interesting to estimate

$$|S_n[f,x] - f(x)|$$

for as many values of α and β as possible.

5. Remarks on Chapter III

In Chapter III a well known analogy between the theories of Fourier series and interpolation is pursued further. The problem of approximating a 2π-periodic function by trigonometric polynomials led to the application of summability methods to Fourier series. Here a "successful" summability method is associated with Lagrange interpolation in order to approximate algebraic functions.

If $T_j(x) = \cos(j \arccos x)$ for $-1 \le x \le 1$, then

$$L_n[f,x] = \sum_{k=0}^{n-1} c_k(f)\, T_k(x)$$

is a representation of the Lagrange interpolation polynomial of degree n-1 on the zeros of $T_n(x)$. The constants $c_k(f)$ are determined by the values of f at these zeros. Motivated by techniques from Fourier series we define

$$\Lambda_n[f,x] = \sum_{k=0}^{n-1} \lambda_k\, c_k(f)\, T_k(x).$$

By a suitable choice of (λ_k) it is shown in Chapter III that $\Lambda_n[f,x]$ provides a uniform approximation of $f(x)$ on $[-1,1]$. It is this idea which is richer than any other in this work in problems for further research.

First there is the perennial problem of what happens on nodes other than the zeros of $T_n(x)$. Also the order of approximation given by the theorem is not the best

possible. This can be seen from the consideration of results in Chapter IV. So we must seek additional restrictions on the coefficients (λ_k) to be able to obtain lower bounds for the error. Finally the result should initiate research into other summability methods applied to Lagrange interpolation.

6. Remarks on Chapters IV and V

In Section 2 an important theorem of A. F. Timan is mentioned. He proved that there is a positive constant A such that, if f(x) is continuous on [-1,1] then, for each natural number n, there is a polynomial P_n of degree n or less such that

$$|P_n(x) - f(x)| \le A[\omega(\frac{\sqrt{(1-x^2)}}{n}) + \omega(\frac{1}{n^2})]$$

for each x in the interval [-1,1]. Later S. A. Teljakovskii [25] showed that Timan's inequality may be replaced by

$$|P_n(x) - f(x)| \le B\, \omega(\frac{\sqrt{(1-x^2)}}{n})$$

for a suitable constant B.

There are two important points about Timan's proof. First the polynomial $P_n(x)$ which he found was of the form

$$P_n(x) \equiv P_n(\cos \Theta) = \int_0^\pi f(\cos y)[T_n(\Theta+y) + T_n(\Theta-y)]dy$$

where T_n was a trigonometric cosine polynomial. Consequently in order to calculate the approximating polynomial he had to know the value of f itself almost everywhere. Second P_n was actually a polynomial of degree 2n-2 rather than n.

In regard to this first point P. L. Butzer [4] asked in 1964 whether it was possible to prove Jackson's Theorem 2 by using a polynomial which is "almost" interpolatory in the sense that it is determined by the values of f at a finite number of nodes. Of course one can ask for the same type of proof for the theorems of Timan and Teljakovskii. One could also demand that the polynomial P_n actually equals the function at the nodes: then we would describe the polynomial as interpolatory.

Since 1964 several answers to Butzer's question have been provided. They are summarized in the table below. But more importantly in each case the construction of the polynomial P_n has been very complicated. In Chapters IV and V a more suitable candidate for P_n is exhibited.

We choose the operator described in Chapter III and consider the special case where $\lambda_k = \cos \frac{k\pi}{2n}$. This particular operator was first discussed by G. Grünwald [12] and so we denote $\Lambda_n[f,x] \equiv \Lambda_n[f,\cos\theta]$ by $G_n[f,\theta]$. Now $G_n[f,\theta]$ has a very nice representation. If $L_n[f,x] = L_n[f,\cos\theta]$, which will be written as $L_n[f,\theta]$, is the Lagrange interpolation polynomial of $f(x)$ on the zeros of $T_n(x) = \cos n\theta$ then

$$G_n[f,\theta] = \frac{1}{2}\left(L_n[f,\theta + \tfrac{\pi}{2n}] + L_n[f,\theta - \tfrac{\pi}{2n}]\right).$$

So the degree of G_n is indeed n-1 and the construction of $G_n(f)$ is very simple. In Chapter IV we see that an estimate of Timan's type can be obtained for the difference $|G_n[f,\theta] - f(\cos\theta)|$. A suitable adjustment to the operator

G_n gives a new operator H_n which is used in Chapter 5 to give a new proof of the theorem of Teljakovskii.

There are a few interesting remaining problems associated with this operator G_n. First there is the saturation problem. It can be easily shown that if f is a linear function then $||G_n(f) - f|| = 0(\frac{1}{n^2})$. I can also show that f is a constant function if and only if $||G_n(f) - f|| = 0(\frac{1}{n^2})$. So perhaps we could identify linear functions by the size of $|G_n[f,x] - f(x)|$. This would not be an easy problem as saturation problems tend to be much more difficult for the class $C([-1,1])$ than $C([0,2\pi])$.

Also it would be interesting to consider similar operators on different nodes. For example, a much more complicated case would arise if we considered the nodes as the zeros of $(1-x^2)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of degree n.

Lastly, the iterates of the operator G_n may well give rise to new interesting theorems just as the iterates of the Bernstein polynomials have.

Author	Nodes	# of Nodes	Interval of Convergence	Degree of Poly.	Nature of Process	Type of Estimate
Freud [8]	$Z(T_n(x))$ in $\|x\|\le\frac{1}{2}$	$\sim\frac{n}{3}$	$\|x\|\le\frac{1}{2}$	4n-3	almost inter-polatory	Jackson
M. Sallay [19]	$Z(P_n(x))$ in $\|x\|\le\frac{1}{2}$		$\|x\|\le\frac{1}{2}$	4n-3	"	Jackson
Saxena [20]	$Z(U_n(x)(1-x^2))$	n+2	$\|x\|\le 1$	4n-2	"	Jackson
Vertesi [28]	$Z(T_n(x)(1-x^2))$	n+2	$\|x\|\le 1$	4n-2	"	Jackson
Mathur [17]	$Z(P_n^{(-\frac{1}{2},\frac{1}{2})}(1-x^2))$	n+2	$\|x\|\le 1$	4n-2	"	Jackson
Freud-Vertesi [10]	$Z(T_n(x)(1-x^2))$	n+2	$\|x\|\le 1$	4n-2	"	Timan
Vertesi-Kis [29]	$Z((1-x)P_n^{(-\frac{1}{2},\frac{1}{2})})$	n+1	$\|x\|\le 1$	4n-4	inter-polatory	Timan
Saxena [21]	$Z((1-x^2)U_n(x))$	n+2	$\|x\|\le 1$	4n-2	almost inter-polatory	Teljak-owskii
Saxena [22]	$Z((1-x^2)T_n(x))$	n+2	$\|x\|\le 1$	4n-2	"	Teljak-owskii

Author	Nodes	# of Nodes	Interval of Convergence	Degree of Poly.	Nature of Process	Type of Estimate
Freud-Sharma [9]	$Z(P_n^{(\alpha,\beta)}(x))(1-x^2)$	n+2	$\|x\| \le 1$	n(1+c) arbitrary	interpolatory (Theo.2)	Timan
Freud-Sharma [9]	$Z(P_n^{(\alpha,\beta)}(x)(1-x^2))$	n+2	$\|x\| \le 1$	"	almost interpolatory	Timan
Chapter 4	$Z(T_n(x))$	n	$\|x\| \le 1$	n-1	"	Timan
Chapter 5	$Z((1-x^2)T_n(x))$	n+2	$\|x\| \le 1$	n-1	"	Teljakowskii

CHAPTER II

ON A THEOREM OF E. EGERVÁRY AND P. TURÁN ON THE STABILITY OF INTERPOLATION

1. Introduction

E. Egerváry and P. Turán [6] have introduced the interpolatory polynomial

$$R_n(f,x) = f(1)\,\frac{1+x}{z}\,P_n{}^2(x) + f(-1)\,\frac{1-x}{2}\,P_n{}^2(x) + \tag{1.1}$$

$$+ \sum_{k=1}^{n} f(x_k)\,\frac{1-x^2}{1-x_k{}^2}\left(\frac{P_n(x)}{(x-x_k)P_n'(x_k)}\right)^2 .$$

Here

$$1 = x_0 > x_1 > \dots > x_n > x_{n+1} = -1 \tag{1.2}$$

are the simple zeros of $(1-x^2)P_n(x)$ and $P_n(x)$ is the Legendre polynomial of degree n. (By writing $P_n(x)$ rather than $P_{n-2}(x)$ in (1.1) I have made a slight change in the notation used in [6].) This process satisfies

$$R_n(f,x_i) = f(x_i) \quad \text{for } i = 0,1,\dots,n+1$$

$$\text{and } R_n'(f,x_i) = 0 \qquad \text{for } i = 1,2,\dots,n$$

and is the most economical stable interpolation process. Recently S. Karlin and W. J. Studden ([14], Chap. 10) have given an extensive discussion of this and similar processes and their relevance to the theory of experimental design.

Theorem 1. (E. Egerváry and P. Turán). The sequence of interpolatory polynomials $R_n(f,x)$ converges uniformily in $[-1,1]$ to $f(x)$ as $n \to \infty$ whenever $f(x)$ is continuous in $-1 \le x \le 1$.

The aim of this chapter is to determine the rate of convergence of $R_n(f,x)$ in terms of the arithmetic means of the sequence $(\omega(\frac{1}{n}))$. Essential to our proof is an important lemma of R. Bojanic [2].

Let $\Omega(t)$ be an increasing, subadditive, continuous function defined for non-negative values of t such that $\Omega(0) = 0$. Let $C_M(\Omega)$ be the class of continuous functions on $[-1,1]$ defined by

$$f \in C_M(\Omega) \quad \text{if and only if} \quad \omega(h) \le M\,\Omega(h)$$

for all $h \ge 0$. Equivalently, $f \in C_M(\Omega)$ if and only if

$$|f(x) - f(y)| \le M\,\Omega(|x-y|)$$

for all $x,y \in [-1,1]$.

We shall prove the following result.

Theorem 2. There exist constants c_1 and c_2 $(0 < c_1 < c_2 < \infty)$ such that for $n \ge 5$,

$$\frac{c_1 M}{n} \sum_{r=5}^{n} \Omega(\tfrac{1}{r}) \le \sup_{f \in C_M(\Omega)} ||R_n(f) - f|| \le \frac{c_2 M}{n} \sum_{r=1}^{n} \Omega(\tfrac{1}{r}).$$

2. Preliminaries

It is well known that

$$\Omega(\lambda t) \le (\lambda+1)\Omega(t) \quad \text{for all } \lambda \ge 0,\ t \ge 0, \tag{2.1}$$

$$0 < t_1 < t_2 \Rightarrow 2\,\frac{\Omega(t_1)}{t_1} \ge \frac{\Omega(t_2)}{t_2}. \tag{2.2}$$

We shall also need the following results concerning Legendre polynomials which can be found in the treatise by G. Szego [24].

For $-1 \le x \le 1$ and $n = 1,2,3,\ldots$.

$$\sum_{k=1}^{n} \frac{1-x^2}{1-x_k^2} \left(\frac{P_n(x)}{(x-x_k)P_n'(x_k)} \right)^2 \equiv 1 - P_n^2(x) \le 1, \tag{2.3}$$

$$(1 - x^2) \quad |P_n(x)| < \frac{2}{\pi n} \quad , \tag{2.4}$$

and

$$(1 - x^2)(P_n'(x))^2 \le n^2. \tag{2.5}$$

Recalling the definition of the x_k's in (1.2), we have [see [24]]

$$|P_n'(x_k)| \sim k^{-3/2}n^2, \; k = 1,2,\ldots,[\tfrac{n}{2}], \tag{2.6}$$

$$|P_n'(x_k)| \sim (n + 1 - k)^{-3/2}n^2, \; k = [\tfrac{n}{2}] + 1,\ldots,n, \tag{2.7}$$

$$(1 - x_k^2) > \frac{(k - 1/2)^2}{(n + 1/2)^2} \quad k = 1,2,\ldots,[\tfrac{n}{2}], \tag{2.8}$$

$$(1 - x_k^2) \quad \frac{(n - k + 1/2)^2}{(n + 1/2)^2} \quad k = [\tfrac{n}{2}] + 1,\ldots,n. \tag{2.9}$$

$$\frac{(k - 1/2)\pi}{(n + 1/2)} < \theta_k < \frac{k\pi}{(n + 1/2)} \quad k = 1,2,\ldots,n,$$

$$x_k = \cos \theta_k. \tag{2.10}$$

3. <u>Estimates</u>

<u>Lemma 1</u>. (R. Bojanic [2].) Let $\Omega(t)$ be a modulus of continuity. Then, for $m \ge s \ge 2$, we have

$$\frac{\pi}{m} \int_{\pi/m}^{s\pi/m} \frac{\Omega(t)}{t^2}\, dt \le \sum_{r=1}^{s-1} \frac{1}{r^2} \; \Omega\left(\frac{(r+1)\pi}{m}\right)$$

$$\le \frac{8\pi}{m} \int_{\pi/m}^{s\pi/m} \frac{\Omega(t)}{t^2}\, dt.$$

Lemma 2. Let $-1 \le x \le +1$ and let x_j be that zero of $P_n(x)$ which is nearest to x. Then we have

$$\frac{(1-x^2)P_n^{\,2}(x)}{(1-x_k^{\,2})(P_n'(x_k))^2(x-x_k)^2} \le \frac{c}{r^2} \quad \text{for } k = j \pm r,$$

$$1 \le r \le n. \tag{3.1}$$

Proof. Let $x = \cos\Theta$, $x_k = \cos\Theta_k$. By using the definition of j and (2.10) it follows that

$$\frac{1}{\sin^2 \frac{\Theta-\Theta_k}{2}} \le \frac{(n+1/2)^2}{(r-1/2)^2} .$$

Since

$$\sin\Theta \le \sin\Theta + \sin\Theta_k \le 2\sin\frac{\Theta+\Theta_k}{2} ,$$

and

$$\sin\Theta_k \le \sin\Theta_k + \sin\Theta \le 2\sin\frac{\Theta+\Theta_k}{2} ,$$

we have on using (2.8)

$$\frac{\sin\Theta}{(\cos\Theta-\cos\Theta_k)^2} = \frac{1}{4\sin^2\frac{\Theta-\Theta_k}{2}} \quad \frac{\sin\Theta}{\sin^2\frac{\Theta-\Theta_k}{2}} ,$$

$$\le \frac{(n+\frac{1}{2})^2}{(r-\frac{1}{2})^2} \quad \frac{1}{\sin\Theta_k} \le$$

$$\le \frac{(n+\frac{1}{2})^3}{(r-\frac{1}{2})^2(k-\frac{1}{2})} \tag{3.2}$$

on using (2.4) - (2.9) we have

$$\frac{(1-x^2)^{1/2}\,P_n^{\,2}(x)}{(1-x_k^{\,2})(P_n'(x_k))^2} \le \frac{2}{\pi n}\,\frac{k}{n^2} = \frac{2k}{\pi n^3} .$$

Now on using (3.2) and the above result we have (3.1).

Lemma 3. (J. Balázs and P. Turán). For the relative extrema of $P_{n-1}(x)$ in $0 < x < 1$ we have the estimation $(n = 4,6,8,\ldots)$

$$|P_{n-1}(\xi_\upsilon)| \geq \frac{1}{\sqrt{(8\pi\upsilon)}}, \quad \upsilon = 2,3,\ldots [\tfrac{n}{2}]$$

$$\geq \frac{1}{\sqrt{(8\pi(n-\upsilon))}} \quad \upsilon = [\tfrac{n}{2}] + 1,\ldots\, n - 1.$$

Further, a similar estimate holds for odd values of n.

For the proof we refer to Lemma 2.1 and corresponding remarks at the end of page 202 in [1]. In this case we need Lemma 3 for one particular value of ξ_υ. Let ξ be the first positive zero of $P_n'(x)$ in $0 < x < 1$. Then from above it follows that

$$|P_n(\xi)| \geq \frac{c_4}{\sqrt{n}} \quad \text{with } c_4 = \frac{1}{2\sqrt{\pi}}\ . \tag{3.3}$$

Lemma 4. Let ξ be the first positive zero of $P_n'(x)$ in (0,1). Then, for $n \geq 5$,

$$A \equiv \sum_{k=1}^{n} \frac{1 - \xi^2}{1 - x_k^2} \left(\frac{P_n(\xi)}{(\xi - x_k) P_n'(x_k)} \right)^2 \Omega(|\xi - x_k|)$$

$$\geq \frac{c_2}{n} \sum_{r=5}^{n} \Omega(\tfrac{1}{r})\ .$$

Proof. Let $\xi = \cos\eta$. Then for $k \leq [\frac{n}{2}] - 1$ it follows from (2.10) that

$$|\xi - x_k| \leq \eta - \theta_k \leq \pi/2 - \theta_k$$

$$\leq \frac{(\frac{n}{2} - k + \frac{3}{4})\pi}{n}\ . \tag{3.4}$$

Hence, by (2.2), (2.6), (2.10), (3.3), and (3.4)

$$A \geq \frac{c_5}{n} \sum_{k=1}^{[\frac{n}{2}]-1} \frac{k}{\frac{n}{2} - k + \frac{3}{4})^2} \left(\frac{(\frac{n}{2} - k + \frac{3}{4})\pi}{n} \right)$$

$$\geq c_6 \sum_{k=[\frac{n}{4}]}^{[\frac{n}{2}]-1} \frac{\Omega\ \frac{(\frac{n}{2} - k + \frac{3}{4})\pi}{n}}{\frac{n}{2} - k + \frac{3}{4})^2} \quad .$$

Now by writing

$$[\tfrac{n}{2}] - k = r$$

and using the properties of Ω we have

$$A \geq c_7 \sum_{r=1}^{[\frac{n}{2}]-[\frac{n}{4}]} \frac{\Omega\ \frac{(j+1)\pi}{n}}{j^2} \quad . \tag{3.5}$$

We now let $m = n$, $s = [\frac{n}{2}] - [\frac{n}{4}] + 1$ and apply Lemma 1 to (3.5) observing that $\frac{s}{n} > \frac{1}{4}$. Hence,

$$A \geq \frac{c_8}{n} \int_{\frac{\pi}{n}}^{\frac{s\pi}{n}} \frac{\Omega(t)}{t^2}\, dt$$

$$\geq \frac{c_8}{n} \int_{\frac{\pi}{n}}^{\frac{\pi}{4}} \frac{\Omega(t)}{t^2}\, dt$$

$$= \frac{c_2}{n} \int_4^n \Omega(\tfrac{\pi}{t})\, dt$$

$$\geq \frac{c_2}{n} \int_4^n \Omega(\tfrac{1}{t})\, dt \geq \frac{c_2}{n} \sum_{r=5}^{n} \Omega(\tfrac{1}{r}) \quad .$$

4. Proof of Theorem 2.

Let $f \in C_M(\Omega)$. (4.1)

From (1.1) and (2.3) we have

$$|R_n(f,x) - f(x)|$$

$$\leq \{\frac{(1 + x)}{2} |f(1) - f(x)| + \frac{(1 - x)}{2} |f(-1) - f(x)|\}P_n{}^2(x) +$$

$$+ \sum_{k=1}^{n} \frac{1 - x^2}{1 - x_k{}^2} \left(\frac{P_n(x)}{P_n{}'(x_k)(x - x_k)} \right)^2 |f(x_k) - f(x)|$$

$$= S_1 + S_2. \tag{4.2}$$

We shall estimate each of the sums S_1 and S_2 separately. By (4.1), (2.1), (2.3), (2.4) and the monotonicity of Ω

$$S_1 \leq M[\frac{(1 + x)}{2} \Omega(1 - x) + \frac{(1 - x)}{2} \Omega(1 + x)]P_n{}^2(x)$$

$$\leq M \Omega(\tfrac{1}{n})[n(1 - x^2)P_n{}^2(x) + P_n{}^2(x)]$$

$$\leq M (\frac{2 + \pi}{\pi}) \Omega(\tfrac{1}{n}) .$$

Hence,

$$S_1 \leq \frac{2M}{n} \sum_{r=1}^{n} \Omega(\tfrac{1}{r}) . \tag{4.3}$$

Choose j such that $1 \leq j \leq n$ and $|x - x_j| \leq |x - x_k|$, $k= 1,2,...,n$. Now, since $f \in C_M(\Omega)$,

$$S_2 \leq M \sum_{k=1}^{n} \frac{1 - x^2}{1 - x_k{}^2} \left(\frac{P_n(x)}{P_n{}'(x_k)(x - x_k)} \right)^2 \Omega(|x_k - x|)$$

$$S_2 \leq M \sum_{k=1}^{n}{}' + M\, \upsilon_j \tag{4.4}$$

where $\sum_{k=1}^{n}{}'$ signifies that the jth term, υ_j, has been omitted.

Before estimating $\sum_{k=1}^{n}{}'$, let us note that for $k = j \pm r$, $r \geq 1$,

$$\Omega(|x_k - x|) = \Omega(|\cos \Theta_k - \cos \Theta|)$$
$$\leq \Omega(|\Theta_k - \Theta|).$$

Thus, by (2.10), and for $k = j \pm r$

$$\Omega(|x_k - x|) \leq c_9 \, \Omega\left(\frac{(r + \frac{1}{2})\pi}{n + \frac{1}{2}}\right) \quad . \tag{4.5}$$

Hence by (3.1) and (4.5) and the property (2.1) of Ω,

$$M \sum_{k=1}^{n}{}' \leq M \, c_{10} \sum_{r=1}^{n-1} \frac{1}{r^2} \, \Omega\left(\frac{(r + 1)\pi}{2n} \right).$$

Let us choose $m = 2n$ and $s = n$ in Lemma 1 and apply it to (4.3). Then we obtain,

$$M \sum_{k=1}^{n}{}' \leq \frac{8\pi c_{10} M}{n} \int_{\frac{\pi}{2n}}^{\pi} \frac{\Omega(t)}{t^2} \, dt$$
$$\leq \frac{c_{10} M}{n} \int_{1}^{2n} \Omega\left(\frac{\pi}{t}\right) dt$$
$$\leq \frac{4c_{10} M}{n} \int_{1}^{2n} \Omega\left(\frac{1}{t}\right) dt \; .$$

Hence

$$M \sum_{k=1}^{n}{}' \leq \frac{c_{11}}{n} \sum_{r=1}^{n} \Omega\left(\frac{1}{r}\right) \; . \tag{4.6}$$

We now turn to estimating υ_j. By (2.10), (2.3), and the monotonicity of Ω, it follows that

$$M \, \upsilon_j = M \, \frac{(1 - x)^2}{1 - x_j^2} \left(\frac{P_n(x)}{(x - x_j) P_n'(x_j)} \right)^2 \Omega(|x - x_j|)$$

$$\leq M\ \Omega(|\theta - \theta_j|).$$

Since $|\theta - \theta_j| < \frac{2}{n}$ we have

$$M\ \upsilon_j \leq 2M\ \Omega(\tfrac{1}{n}) \leq \frac{2M}{n} \sum_{r=1}^{n} \Omega(\tfrac{1}{r}) \ . \tag{4.7}$$

Thus from (4.2), (4.3), (4.4), (4.6), and (4.7), the upper estimate in Theorem 2 follows.

To prove the lower estimate in Theorem 2, consider the function

$$g(x) = M\ \Omega(|x - \xi|)$$

where ξ is as in Lemma 4. It is easily verified that $g \in C_M(\Omega)$.

Hence $\sup_{f \in C_M(\Omega)} ||R_n(f) - f||$

$$\geq ||R_n(g) - g||$$

$$\geq |R_n(g,\xi) - g(\xi)|$$

$$= R_n(g,\xi) \equiv A \cdot$$

The lower estimate in Theorem 2 now follows from Lemma 4.

CHAPTER III

ON THE SUMMABILITY OF LAGRANGE INTERPOLATION

1. Introduction

Let

$$B \equiv \begin{array}{llll} x_1^{(1)} & & & \\ x_1^{(2)}, & x_2^{(2)} & & \\ \vdots & \vdots & & \\ x_1^{(n)}, & x_2^{(n)}, & \ldots, & x_n^{(n)} \\ \vdots & \vdots & & \vdots \;\ddots \end{array}$$

be an aggregate of points such that, for each n,

$$1 \geq x_1^{(n)} > x_2^{(n)} > \ldots > x_n^{(n)} \geq -1.$$

For any continuous function $f(x)$ with domain $[-1,1]$ we define the nth Lagrange interpolation polynomial of $f(x)$ with respect to B to be that unique polynomial of degree at most $n - 1$ which assumes the values $f(x_1^{(n)}),\ldots, f(x_n^{(n)})$ at $x_1^{(n)}, x_2^{(n)},\ldots, x_n^{(n)}$, respectively.

Here we shall consider the case where the points $x_k^{(n)}$ $(k = 1,\ldots, n)$ are the zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$: that is

$$x_k^{(n)} = \cos(2k - 1)\pi/2n, \quad k = 1,\ldots, n.$$

These interpolation polynomials are given by

$$L_n(f,x) = \sum_{k=1}^{n} f(x_k^{(n)})T_n(x)/((x - x_k^{(n)})T'(x_k^{(n)})).$$

Since, as it is easy to verify,

$$T_n(x)/(x - x_k^{(n)})T_n'(x_k^{(n)}) = (1/n)(1 + 2\sum_{r=1}^{n-1} T_r(x)T_r(x_k^{(n)})),$$

the $L_n(f)$ can be expressed as follows:

$$L_n(f,x) = \sum_{r=0}^{n-1} c_r(f)T_r(x),$$

where

$$c_0(f) = 1/n \sum_{k=1}^{n} f(x_k^{(n)})$$

and

$$c_r(f) = 2/n \sum_{k=1}^{n} f(x_k^{(n)})T_r(x_k^{(n)}), \quad r = 1,\dots, n-1.$$

The properties of this sequence of polynomials are sometimes similar to those of the partial sums of the Fourier series of an integrable 2π-periodic function. Therefore, as in the theory of Fourier series, it is natural to consider summability methods which would sum the sequence $(L_n(f))$ to f for a large class of functions.

We shall consider summability methods

$$\Lambda_n(f,x) = \sum_{r=0}^{n-1} \lambda_r^{(n)} c_r(f)T_r(x), \tag{1.1}$$

which arise from a triangular matrix $(\lambda_k^{(n)})k = 0,1,\dots,n-1;$ $n = 1,2,\dots$.

It is easy to see that

$$\Lambda_n(f,x) = \sum_{k=1}^{n} f(x_k^{(n)})A_{k,n}(x),$$

where

$$A_{k,n}(x) = (1/n)(1 + 2\sum_{r=1}^{n-1} \lambda_r^{(n)} T_r(x)T_r(x_k^{(n)})).$$

Here we prove the following theorem.

Theorem. Let the matrix of coefficients $(\lambda_j^{(n)})$ satisfy

$$\lambda_0^{(n)} = 1; \quad \lambda_j^{(n)} = 0 \text{ if } j \geq n; \quad \lambda_{n-1}^{(n)} = 0(1/n) \tag{1.2}$$

and either

$$|\lambda_{j+1}^{(n)} - 2\lambda_j^{(n)} + \lambda_{j-1}^{(n)}| = 0(1/n^2), \quad j = 1,\ldots, n-1 \tag{1.3}$$

or

$$1 - \lambda_1^{(n)} = 0(1/n), \quad \lambda_{j+1}^{(n)} - 2\lambda_j^{(n)} + \lambda_{j-1}^{(n)} \geq 0 \quad j = 1,\ldots,n-1. \tag{1.4}$$

Then

$$||\Lambda_n(f) - f|| \leq c_1/n \sum_{r=1}^{n} \omega(1/r),$$

for $f \in C[-1,1]$ having modulus of continuity $\omega(\delta)$. The constant c_1 (and elsewhere $c_2, c_3, \ldots$) is positive and independent of n and f.

Now we give some choice of λ_j's which satisfy the above requirement.

(a) $$\lambda_j^{(n)} = \frac{(n-j)^m}{(n-j)^m + j^m} \quad j = 0,1,\ldots, n-1$$
$$= 0 \quad j \geq n;$$

(b) $$\lambda_j^{(n)} = \frac{(n-j)^m}{n^m} \quad j = 0,1,\ldots, n$$
$$= 0 \quad j \geq n.$$

2. Preliminaries

If $f(x) \equiv 1$ then

$$c_0(f) = 1$$

and, for $r = 1,2,\ldots, n-1$,

$$c_r(f) = 2/n \sum_{k=1}^{n} T_r(x_k^{(n)})$$

$$= 2/n \sum_{k=1}^{n} \cos(((2k - 1)r/2n)\pi)$$

$$= 0.$$

So by (1.1) and (1.2)

$$\Lambda_n(1,x) \equiv 1,$$

and, therefore,

$$|\Lambda_n(f,x) - f(x)| \le \sum_{k=1}^{n} |f(x_k^{(n)}) - f(x)||A_{kn}(x)|$$

$$\le \sum_{k=1}^{n} \omega(|x_k^{(n)} - x|)|A_{kn}(x)|.$$

Let $x = \cos\theta$, $x_k^{(n)} = \cos\theta_k^{(n)}$, $k = 1,\ldots, n$. We have then

$$|\Lambda_n(f,x) - f(x)| \le \sum_{k=1}^{n} \omega(|\theta_k^{(n)} - \theta|)|P_{k,n}(\theta)|, \tag{2.1}$$

where

$$P_{k,n}(\theta) = A_{k,n}(\cos\theta)$$

$$= (1/n) + (2/n) \sum_{r=1}^{n-1} \lambda_r^{(n)} \cos r\theta \cos r\theta_k^{(n)}.$$

To prove the theorem we need some preliminary notation and estimates.

We denote the Fejér kernel by

$$t_j(\theta) = 1 + \frac{2}{j} \sum_{i=1}^{j-1} (j - i)\cos i\theta$$

$$= \frac{1}{j}\left(\frac{\sin j\theta/2}{\sin \theta/2}\right)^2 \quad \text{for } j = 2, 3, \ldots, n$$

and $t_1(\theta) \equiv 1$.

Associated with this kernel we introduce

$$\tau_{j,k}(\theta) = \frac{1}{2}(t_j(\theta + \theta_k^{(n)}) + t_j(\theta - \theta_k^{(n)})).$$

It is easy to verify that

$(j+1)\tau_{j+1,k}(\Theta) - 2j\tau_{j,k}(\Theta) + (j-1)\tau_{j-1,k}(\Theta) = 2\cos j\Theta \cos j\Theta_k^{(n)}$.

Using this relation we obtain

$$P_{k,n}(\Theta) = 1/n \sum_{r=1}^{n-1} (\lambda_{r-1}^{(n)} - 2\lambda_r^{(n)} + \lambda_{r+1}^{(n)}) r\tau_{r,k}(\Theta) + \lambda_{n-1}^{(n)}\tau_{n,k}(\Theta). \quad (2.2)$$

If there is no confusion we shall write A_k, P_k, λ_k, Θ_k, for $A_{k,n}$, $P_{k,n}$, $\lambda_k^{(n)}$, $\Theta_k^{(n)}$.

Naturally enough, we shall require the following lemma.

Lemma 1. Under the hypotheses (1.2), (1.3) or (1.2), (1.4) we have

$$\sum_{k=1}^{n} |A_k(x)| = \sum_{k=1}^{n} |P_k(\Theta)| = 0(1).$$

Proof. Let (1.2) and (1.3) hold. That is,

$$\lambda_0 = 1, \quad \lambda_j = 0 \quad \text{if } j \geq n, \quad \lambda_{n-1} = 0(1/n)$$

and

$$|\lambda_{j+1} - 2\lambda_j + \lambda_{j-1}| = 0(1/n^2) \quad j = 1,\ldots, n-1.$$

Then by these hypotheses and (2.2) we have

$$\sum_{k=1}^{n} |P_k(\Theta)| \leq \sum_{k=1}^{n} 1/n\Big(\sum_{j=1}^{n-1} |\lambda_{j+1} - 2\lambda j + \lambda_{j-1}| j\tau_{j,k}(\Theta)\Big) + \sum_{k=1}^{n} |\lambda_{n-1}|\tau_{n,k}(\Theta)$$

$$= 1/n \sum_{j=1}^{n-1} 0(1/n^2), jn + 0(1/n)n$$

$$= 0(1).$$

Alternatively, let (1.2) and (1.4) hold. That is

$$\lambda_0 = 1, \quad \lambda_j = 0 \quad \text{if } j \geq n, \quad \lambda_{n-1} = 0(1/n)$$

and

$$1 - \lambda_1 = 0(1/n), \quad \lambda_{j+1} - 2\lambda j + \lambda_{j-1} \geq 0, \quad j = 1,\ldots,n-1.$$

Then we have

$$\sum_{k=1}^{n} |P_k(\theta)| \le \sum_{j=1}^{n-1} (\lambda_{j+1} - 2\lambda_j + \lambda_{j-1})j + nO(1/n) = O(1).$$

Lemma 2. Let $\theta \neq \theta_k$. Then for $1 \le k \le n$ and $1 \le r \le n$,

$$\tau_{r,k}(\theta) \le \pi^2/r(\theta - \theta_k)^2.$$

Proof. By definition,

$$\tau_{r,k}(\theta) = \frac{1}{2r}\left(\frac{\sin^2(r(\theta + \theta_k))/2}{\sin(\theta + \theta_k)/2} + \frac{\sin^2(r(\theta - \theta_k))/2}{\sin^2(\theta - \theta_k)/2} \right) \quad (2.3)$$

Also

$$\sin \frac{\theta + \theta_k}{2} = \sin \frac{\theta}{2} \cos \frac{\theta_k}{2} + \cos \frac{\theta}{2} \sin \frac{\theta_k}{2},$$

and, hence,

$$\left|\sin \frac{\theta + \theta_k}{2}\right| \ge \left|\sin \frac{\theta}{2} \sin \frac{\theta_k}{2} - \cos \frac{\theta}{2} \sin \frac{\theta_k}{2}\right| = \sin \left|\frac{\theta - \theta_k}{2}\right|. \quad (2.4)$$

Then the lemma follows from (2.3), (2.4), and Jordan's inequality, namely,

$$|\sin x| \ge 2/\pi |x| \qquad \text{if } 0 \le |x| \le \pi/2.$$

Lemma 3. Let $\theta \in [0,\pi]$, and let $\theta_j = ((2j - 1)\pi)/2n$ be the node nearest to θ. Then

$$\sum_{k=1}^{j-1} (\omega(|\theta_k - \theta|))/(\theta_k - \theta)^2 \le c_2 n \sum_{r=1}^{n} \omega(1/r)$$

and

$$\sum_{k=j+1}^{n} (\omega(|\theta_k - \theta|))/(\theta_k - \theta)^2 \le c_3 n \sum_{r=1}^{n} \omega(1/r).$$

(If $j = 1$ or n then only one of these inequalities holds.)

Proof. This lemma is contained implicitly in a paper given by Bojanic [2].

3. Proof of the Theorem

We can now prove the theorem. Let j be as in Lemma 3. By (2.1),

$$|\Lambda_n(f,x) - f(x)| \le \sum_{k=1}^{j-1} \omega(|\Theta_k - \Theta|)|P_k(\Theta)| + \omega(|\Theta_j - \Theta|)|P_j(\Theta)|$$

$$+ \sum_{k=j+1}^{n} \omega(|\Theta_k - \Theta)|P_k(\Theta)|.$$

As remarked before, the first or last sum may not appear in some cases.

By our choice of Θ_j and Lemma 1,

$$\omega(|\Theta_j - \Theta|)|P_j(\Theta)| \le c_3\omega(1/n)|P_j(\Theta)|$$

$$\le c_4\omega(1/n)$$

$$\le C_5/n \sum_{r=1}^{n} \omega(1/r). \qquad (3.1).$$

Suppose (1.2) and (1.3) hold. Then by these hypotheses and Lemmas 2 and 3,

$$\sum_{k=1}^{j-1} \omega(|\Theta_k - \Theta|)|P_k(\Theta)|$$

$$\le \sum_{k=1}^{j-1} \omega(|\Theta_k - \Theta|)(1/n \sum_{r=1}^{n-1} |\lambda_{r+1} - 2\lambda_r+\lambda_{r-1}|r\tau_{r,k}(\Theta)+|\lambda_{n-1}|\tau_{n,k}(\Theta))$$

$$\le c_6/n^3 \sum_{k=1}^{n-1} r \sum_{k=1}^{j-1} \omega(|\Theta_k - \Theta|)\tau_{r,k}(\Theta)+ \sum_{k=1}^{j-1} |\lambda_{n-1}|\omega(|\Theta_k-\Theta|)\tau_{n,k}(\Theta)$$

$$\le c_7/n^3 \sum_{r=1}^{n-1} \sum_{k=1}^{j-1} (\omega(|\Theta_k-\Theta|)/(\Theta_k-\Theta)^2)+c_8/n \sum_{k=1}^{j-1} \omega(|\Theta_k-\Theta|)\tau_{r,k}(\Theta)$$

$$\le c_9/n \sum_{r=1}^{n} \omega(1/r). \qquad (3.2).$$

A similar estimate is valid for $\Sigma_{k=j+1}^{n}\omega(|\Theta_k-\Theta|)|P_k(\Theta)|$ and so by (3.1) and (3.2) the proof is complete.

It remains to consider the case when (1.2) and (1.4) hold. Since the inequality (3.1) is still valid, it suffices to estimate the sum $\Sigma_{k=1}^{j-1}$ under these conditions.

Now

$$A \equiv \sum_{k=1}^{j-1} \omega(|\Theta - \Theta_k|)|P_k(\Theta)|$$

$$\leq \sum_{k=1}^{j-1} \omega(|\Theta-\Theta_k|)(1/n \sum_{r=1}^{n-1} (\lambda_{r+1}-2\lambda_r+\lambda_{r-1})r\tau_{r,k}(\Theta)+|\lambda_{n-1}|\tau_{n,k}(\Theta))$$

$$= 1/n \sum_{r=1}^{n-1} (\lambda_{r+1} - 2\lambda_r + \lambda_{r-1})r \sum_{k=1}^{j-1} \omega(|\Theta - \Theta_k| \tau_{r,k}$$

$$+ |\lambda_{n-1}| \sum_{k=1}^{j-1} \omega(|\Theta - \Theta|)\tau_{n,k}(\Theta).$$

In using Lemmas 2 and 3 and the hypotheses (1.2) and (1.4).

$$A \leq 1/n \sum_{r=1}^{n-1} (\lambda_{r+1} - 2\lambda_r + \lambda_{r-1}) \sum_{k=1}^{j-1} (\omega(|\Theta - \Theta_k|)/(\Theta - \Theta_k)^2)$$

$$+ c_{10}/n^2 \sum_{k=1}^{j-1} (\omega(|\Theta - \Theta_k|)/(\Theta - \Theta_k)^2)$$

$$\leq c_{11}(\sum_{r=1}^{n} \omega(1/r))(1 - \lambda_1 - \lambda_{n-1}) \leq c_{12}/n \sum_{r=1}^{n} \omega(1/r).$$

Again a similar estimate is valid for $\sum_{k=j+1}^{n}$ and the proof of the theorem is complete.

CHAPTER IV

NEW ESTIMATES FOR G. GRÜNWALD'S CONVERGENCE THEOREM CONCERNING LAGRANGE INTERPOLATION AND A NEW PROOF OF A THEOREM OF A. F. TIMAN

1. Introduction

Let
$$x_k^{(n)} = \cos \frac{(2k-1)\pi}{2n} \quad k = 1,2,\ldots,n$$
$$= \cos \theta_k^{(n)} \quad k = 1,2,\ldots,n$$

and suppose that $f(x)$ is a continuous function defined on $[-1,+1]$. With the convention $x = \cos \theta$ we define $L_n(f,\theta)$ to be the nth Lagrange interpolation polynomial of f. That is, $L_n(f,\theta)$ is the polynomial of degree n-1 such that
$$L_n(f,\theta_k) = f(\cos \theta_k^{(n)}).$$
We may represent $L_n(f)$ as

$$L_n[f,\theta] = \sum_{k=1}^{n} f(x_k^{(n)})\ell_k^{(n)}(\theta),$$

$$\ell_k^{(n)}(\theta) = \frac{\cos n\theta \sin \theta_k^{(n)}(-1)^{k+1}}{n(\cos \theta - \cos \theta_k^{(n)})}, \quad \cos \theta = x.$$

G. Grünwald [12] has defined a new operator by
$$G_n(f,\theta) = \frac{1}{2}(L_n(f,\theta + \tfrac{\pi}{2n}) + L_n(f,\theta - \tfrac{\pi}{2n}))$$
which is analogous to the Bernstein-Rogosinski summability method of a Fourier series. We may represent $G_n(f)$ by

$$G_n(f,\Theta) = \sum_{k=1}^{n} f(x_k^{(n)})A_k(\Theta) \qquad (1.1)$$

where $A_k(\Theta) = \frac{1}{2}(\ell_k(\Theta + \frac{\pi}{2n}) + \ell_k(\Theta - \frac{\pi}{2n}))$. Moreover he proved the following result.

Theorem (G, Grünwald [12])

If $f \in C([-1,1])$ then $\lim_{n\to\infty} G_n(f,\Theta) = f(\cos\Theta)$ and the convergence is uniform in $[-1,1]$.

In this paper we want to prove two qualitative versions of this theorem. More explicitly we shall prove.

Theorem 1. Let $f \in C([-1,1])$ and let $\omega_2(\delta)$ be the second modulus of continuity of f. Then

$$||G_n(f) - f|| = 0(\omega_2(\tfrac{1}{n})).$$

It is interesting to note that S. B. Stechkin (see [27] page 326, problem 7) has obtained similar estimates concerning the Bernstein-Rogosinski summability method of a Fourier series. We employ D. Jackson's fundamental theorem in the proof of Theorem 1.

However without recourse to Jackson's theorem we shall establish a pointwise estimate of the error.

Theorem 2. Let $\omega(\delta)$ be the modulus of continuity of $f \in C([-1,1])$. Then

$$|G_n(f,\Theta) - f(\cos\Theta)| = 0(\omega(\tfrac{\sin\Theta}{n}) + \omega(\tfrac{1}{n^2})).$$

This latter theorem provides a new proof of the fundamental theorem of A. F. Timan ([27], page 262). In this light it is interesting to note that the degree of $G_n(f,\Theta)$ is n-1 and that G_n is not a positive operator.

Throughout this paper $\Theta_k^{(n)}$ will be denoted by Θ_k, $x_k^{(n)}$ by x_k, $\ell_k^{(n)}(\Theta)$ by $\ell_k(\Theta)$ and C will denote universal constants. $(C > 0)$

2. Proof of Theorem 1.

By D. Jackson's theorem there is a polynomial $P_{n-1}(x)$ of degree n-1 such that $||f-P_{n-1}|| = 0(\omega_2(\frac{1}{n}))$. (2.1)

Hence by G. Grünwald's theorem, the uniform boundedness principle for linear operators, the definition of G_n and (2.1),

$$|G_n(f,\Theta) - f(\cos\Theta)|$$

$$\leq |G_n(f-P_{n-1},\Theta)| + |G_n(P_{n-1},\Theta) - f(\cos\Theta)|$$

$$\leq C\,\omega_2(\tfrac{1}{n})+\tfrac{1}{2}|P_{n-1}(\cos(\Theta+\tfrac{\pi}{2n})) + P_{n-1}(\cos(\Theta-\tfrac{\pi}{2n}))-2f(\cos\Theta)|.$$

Now we let $\cos(\Theta + \frac{\pi}{2n}) = h + \cos\Theta = h + x$ and $\cos(\Theta - \frac{\pi}{2n}) = \cos\Theta - k = x - k$.

Then

$$\begin{aligned}|G_n(f,\Theta) - f(\cos\Theta)| \leq c\omega_2(\tfrac{1}{n}) &+ \tfrac{1}{2}|P_{n-1}(x+h) - f(x+h)| \\ &+ \tfrac{1}{2}|P_{n-1}(x-k) - f(x-k)| \\ &+ \tfrac{1}{2}|f(x+h) - f(x-h) - 2f(x)| \\ &+ \tfrac{1}{2}|f(x-k) - f(x-h)|.\end{aligned}$$

By (2.1) and the fact that $h = 0(\frac{1}{n})$, we have

$$\begin{aligned}|G_n(f,\cos\Theta) - f(\cos\Theta)| &= 0(\omega_2(\tfrac{1}{n})) + \tfrac{1}{2}|f(x-k) - f(x-h)| \\ &= 0(\omega_2(\tfrac{1}{n})) + \omega(|h-k|).\end{aligned}$$

But h-k =

$$= [\cos (\Theta + \frac{\pi}{2n}) - \cos \Theta]-[\cos \Theta - \cos(\Theta - \frac{\pi}{2n})]$$

$$= -2 \sin \frac{\pi}{4n} \sin(\Theta + \frac{\pi}{4n}) + 2 \sin \frac{\pi}{4n} \sin(\Theta - \frac{\pi}{2n})]$$

$$= -4 \sin^2 \frac{\pi}{4n} \cos \Theta$$

$$= 0(\frac{1}{n^2}) .$$

And so, using Theorem 9 on page 48 of [16] we have

$$\omega(|h-k|) = 0(\omega(\frac{1}{n^2})) = 0(\omega_2(\frac{1}{n})) .$$

Therefore

$$||G_n(f) - f|| = 0(\omega_2(\frac{1}{n}))$$

and Theorem 1 is proved.

3. Preliminaries.

In this section we shall adopt the following notation:

$$a(k) = \sin \frac{1}{2}(\frac{k\pi}{n} + \Theta)$$

and

$$b(k) = \sin \frac{1}{2}(\frac{k\pi}{n} - \Theta) .$$

Lemma 1. (O. Kis [15]) Let $\frac{(j-1)\pi}{n} < \Theta < \frac{j\pi}{n}$. Then

$$|f(x_k) - f(x)| = 0[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{1}{n^2})] \text{ if } k = j \tag{3.1}$$

$$= 0 [\omega(\frac{i \sin \Theta}{n}) + \omega(\frac{i^2}{n^2})] \tag{3.2}$$

if $1 \le k = j - i < j$

or $j < k = j + i \le n$.

Lemma 2. (O. Kis [15]) Let $\frac{(j-1)\pi}{n} < \Theta < \frac{j\pi}{n}$. Then

$$|f(x_k) - f(x_{k+1})| = 0[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{i}{n^2})] \text{ if } j < k = j + i \le n,$$

$$|f(x_k) - f(x_{k-1})| = 0[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{i}{n^2})] \text{ if } 1 \le k = j - i < j .$$

<u>Lemma 3</u>. If $\frac{(j-1)\pi}{n} < \Theta < \frac{j\pi}{n}$ then

$$|A_k(\Theta)| = 0(1) \text{ if } k = j-1, j, \text{or } j+1, \quad (3.3)$$

$$|A_k(\Theta)| = 0(\frac{1}{i^2}) \text{ if } 1 \le k = j-i < j-1$$

$$\text{or } j+1 < k = j+i \le n, \quad (3.4)$$

and

$$|A_k(\Theta)| = 0(\frac{n \sin \Theta}{i^3}) \text{ if } 1 \le k = j-i<j$$

$$\text{or } j < k = j+i \le n . \quad (3.5)$$

<u>Proof</u>. G. Grünwald [12] has proved that $|A_j(\Theta)| = 0(1)$. Similarly the rest of (3.3) can be shown. It is well known that

$$\ell_k(\Theta) = (-1)^{k+1} \frac{\cos n\Theta \sin \Theta_k}{n(\cos \Theta_k - \cos \Theta)}$$

$$= (-1)^k \frac{\cos n\Theta}{2n} (\cot \frac{\Theta_k - \Theta}{2} + \cot \frac{\Theta_k + \Theta}{2}) .$$

Hence

$$2A_k(\Theta) = \ell_k(\Theta + \frac{\pi}{2n}) + \ell_k(\Theta - \frac{\pi}{2n})$$

$$= (-1)^{k+1} \frac{\sin n\Theta}{2n} [\cot \frac{1}{2}(\frac{(k-1)\pi}{n} - \Theta) + \cot \frac{1}{2}(\frac{k\pi}{n} + \Theta) -$$

$$- \cot \frac{1}{2}(\frac{(k-1)\pi}{n} + \Theta) - \cot \frac{1}{2}(\frac{k\pi}{n} - \Theta)] . \quad (3.6)$$

Then

$$2A_k(\Theta) = (-1)^{k+1} \frac{\sin n\Theta \sin \frac{\pi}{2n}}{2n} [\frac{1}{b(k-1)b(k)} -$$

$$- \frac{1}{a(k-1)a(k)}] . \quad (3.7)$$

By Jordan's inequality we have

$$|b(k)| = |\sin \frac{1}{2}(\frac{k\pi}{n}) - \Theta)|$$

$$\ge \frac{i}{n} \text{ if } j < k = j+i \le n . \quad (3.8)$$

Similarly $|b(k-1)| \geq \frac{ci}{n}, j+1 < k = j+i \leq n.$ (3.9)

Furthermore if $0 \leq \alpha, \beta \leq \pi$ then

$$\sin\tfrac{1}{2}(\alpha+\beta) = \sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta + \cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta$$

$$\geq |\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta - \cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta|$$

$$= \sin\tfrac{1}{2}|\alpha-\beta| .$$

Thus if $j+1 < k = j+i \leq n$

$$|a(k)| = |\sin\tfrac{1}{2}(\Theta + \Theta_k + \tfrac{\pi}{2n})|$$

$$\geq b(k)$$

$$\geq \frac{i}{n} \tag{3.10}$$

and $|a(k-1)| \geq \frac{ci}{n}$. (3.11)

We shall make several uses of these lower estimates.

If we return to (3.7) we now see that

$$A_k(\Theta) = 0(\tfrac{1}{i^2}) \text{ if } j+1 < k+i \leq n$$

which proves part of (3.4). A similar argument proves the rest of (3.4).

From [1] we have the representation

$$A_k(\Theta) = \tfrac{1}{2}(-1)^{k+1}\frac{\sin\Theta_k \sin n\Theta \sin\Theta \sin\frac{\pi}{2n}}{4n\; a(k-1)a(k)b(k-1)b(k)} .$$

Now let us represent $\sin\Theta_k$ as

$$\sin\Theta_k = \sin[\tfrac{1}{2}(\Theta + \Theta + \tfrac{\pi}{2n}) - \tfrac{1}{2}(\Theta - \Theta_k + \tfrac{\pi}{2n})]$$

$$= 0(a(k) + b(k)) . \tag{3.12}$$

Then $$|A_k(\Theta)| \leq \frac{\sin\Theta\sin\frac{\pi}{2n}}{8n}\left(\frac{a(k)+b(k)}{a(k-1)a(k)b(k-1)b(k)}\right) =$$

$$= 0(\frac{n\sin\Theta}{i^3})$$

which proves (3.5).

Lemma 4. If $\frac{(j-1)\pi}{n} < \theta < \frac{j\pi}{n}$ then

$$|A_k(\theta) + A_{k+1}(\theta)| = O(\frac{1}{i^3}) \text{ if } j + 1 < k = j + 1 < n \qquad (3.13)$$

and

$$|A_k(\theta) + A_{k-1}(\theta)| = O(\frac{1}{i^3}) \text{ if } 1 < k = j - i < j - 1 \,. \qquad (3.14)$$

Also we have

$$|A_k(\theta) + A_{k+1}(\theta)| = O(\frac{n \sin \theta}{i^4}) \text{ if } j + 1 < k = j + i < n-1 \quad (3.15)$$

and

$$|A_k(\theta) + A_{k-1}(\theta)| = O(\frac{n \sin \theta}{i^4}) \text{ if } 2 < k = j - i < j - 1 \,. \qquad (3.16)$$

Proof. Let us suppose that $j + 1 < k = j + i < n$. Then by (3.6),

$$A_k(\theta) + A_{k+1}(\theta) = (-1)^{k+1} \frac{\sin n\theta}{2n}[\cot \tfrac{1}{2}(\frac{(k+1)\pi}{n} \theta) -$$

$$- 2 \cot \tfrac{1}{2}(\frac{k\pi}{n} - \theta)] + \cot \tfrac{1}{2}(\frac{(k-1)\pi}{n} - \theta)] +$$

$$+ (-1)^{k+1} \frac{\sin n\theta}{2n} [\cot \tfrac{1}{2}(\frac{(k+1)\pi}{n} + \theta) - 2 \cot \tfrac{1}{2}(\frac{k\pi}{n} + \theta) +$$

$$+ \cot \tfrac{1}{2}(\frac{(k-1)\pi}{n} + \theta)] = (-1)^{k+1}S_1 + (-1)^{k+2}S_2 \text{ say.} \qquad (3.17)$$

Now as in Lemma 3, we have

$$\cot \tfrac{1}{2}(\frac{(k+1)\pi}{n} - \theta) - \cot \tfrac{1}{2}(\frac{k\pi}{n} - \theta) = \frac{-\sin \frac{\pi}{2\pi}}{b(k+1)b(k)}$$

and

$$\cot \tfrac{1}{2}(\frac{(k-1)\pi}{n} - \theta) - \cot \tfrac{1}{2}(\frac{k\pi}{n} - \theta) = \frac{-\sin \frac{\pi}{2n}}{b(k-1)b(k)} \,.$$

Hence using inequalities like (3.8),

$$S_1 = \frac{\sin n\theta \sin \frac{\pi}{2n}}{2n\, b(k)} [\frac{1}{b(k-1)} - \frac{1}{b(k+1)}]$$

$$= \frac{\sin n\theta \sin \frac{\pi}{2n} \cos \tfrac{1}{2}(\frac{k\pi}{n} - \theta) \sin \frac{\pi}{2n}}{2n \quad b(k-1) \quad b(k) \; b(k+1)}$$

$$= O(\frac{1}{i^3}) \,.$$

Similarly $S_2 = \dfrac{\sin n\Theta \sin \frac{\pi}{2n} \cos \frac{1}{2}(\frac{k\pi}{n} + \Theta) \sin \frac{\pi}{2n}}{n \quad a(k-1) \quad a(k) \quad a(k+1)}$

$$= 0(\frac{1}{i^3}).$$

Thus (3.13), and likewise (3.14), are shown.

To get the local estimates (3.15) and (3.16) we must go further with the above calculations.

Noting that by (3.17) $|A_k(\Theta) + A_{k+1}(\Theta)| = |S_1 - S_2|$, we now calculate $S_1 - S_2$.

$$S_1 - S_2 = \frac{\sin n\Theta \sin^2 \frac{\pi}{2n}}{2n} \quad \frac{\cos \frac{1}{2}(\frac{k\pi}{n} - \Theta)}{b(k-1)b(k)b(k+1)} -$$

$$- \frac{\cos \frac{1}{2}(\frac{k\pi}{n} + \Theta)}{a(k-1)a(k)a(k+1)} . \tag{3.18}$$

Now $2 \cos \frac{1}{2}(\frac{k\pi}{n} - \Theta)\ a(k-1)\ a(k)\ a(k+1)$

$= 2 \cos \frac{1}{2}(\frac{k\pi}{n} - \Theta) \sin \frac{1}{2}(\frac{k\pi}{n} + \Theta)\ a(k-1)\ a(k+1)$

$= [\sin \frac{k\pi}{n} + \sin \Theta]\ a(k-1)\ a(k+1).$

Similarly

$2 \cos \frac{1}{2}(\frac{k\pi}{n} + \Theta)\ b(k-1)\ b(k)\ b(k+1) =$
$= (\sin \frac{k\pi}{n} - \sin \Theta) b(k-1) b(k+1)$.

Then

$(\sin \frac{k\pi}{n} + \sin \Theta\ a(k-1) a(k+1) - (\sin \frac{k\pi}{n} - \sin \Theta) b(k-1) b(k+1) =$
$= \sin \frac{k\pi}{n} [a(k-1)\ a(k+1) - b(k-1)\ b(k+1)] +$
$+ \sin \Theta\ [a(k-1)\ a(k+1) + b(k-1)\ b(k+1)].$ (3.19)

Further, we have

$$a(k-1)\ a(k+1) - b(k-1)\ b(k+1)$$
$$= \sin \tfrac{1}{2}(\tfrac{(k-1)\pi}{n} + \Theta) \sin \tfrac{1}{2}(\tfrac{(k+1)\pi}{n} + \Theta) -$$
$$- \sin \tfrac{1}{2}(\tfrac{(k-1)\pi}{n} - \Theta) \sin \tfrac{1}{2}(\tfrac{(k+1)\pi}{n} - \Theta)$$
$$= \tfrac{1}{2}[\cos \tfrac{\pi}{n} - \cos(\tfrac{k\pi}{n} + \Theta) - \cos \tfrac{\pi}{n} + \cos(\tfrac{k\pi}{n} - \Theta)]$$
$$= \sin \Theta \sin \tfrac{k\pi}{n} \quad .$$

Hence turning back to (3.19),

$$\sin \tfrac{k\pi}{n}[a(k-1)\ a(k+1) - b(k-1)\ b(k+1)] = \sin \Theta \sin^2 \tfrac{k\pi}{n} . \quad (3.20)$$

If we use (3.19) and (3.20) in (3.18) we have

$$S_1 - S_2 = \frac{\sin n\Theta \sin^2 \frac{\pi}{2n} \sin \Theta}{2n}\left[\frac{a(k-1)a(k+1)+b(k-1)b(k+1)+\sin^2 \frac{k\pi}{n}}{a(k-1)a(k)a(k+1)b(k-1)b(k)b(k+1)}\right]$$

As in (3.8) - (3.11)

$$\frac{a(k-1)a(k+1)}{a(k-1)a(k)a(k+1)b(k\ 1)b(k)b(k+1)} = 0(\frac{n^4}{i^4})$$

and

$$\frac{b(k-1)b(k+1)}{a(k-1)a(k)a(k+1)\ b(k-1)b(k)b(k+1)} = 0(\frac{n^4}{i^4}) .$$

Finally we can obtain the estimate

$$\frac{\sin^2(\frac{k\pi}{n})}{a(k-1)a(k)a(k+1)b(k-1)b(k)b(k+1)} = 0(\frac{n \sin \Theta}{i^4})$$

by using the same technique as in (3.12). So we have

$$S_1 - S_2 = 0(\frac{n \sin \Theta}{i^4}) .$$

Therefore

$$|A_k(\Theta) + A_{k+1}(\Theta)| = |S_1 - S_2|$$
$$= 0\ (\frac{n \sin \Theta}{i^4})$$

if $j + 1 < k = j + i < n - 1$ and (3.15) has been established. Similarly (3.16) can be shown.

4. Proof of Theorem 2.

By (1.1),

$$G_n(f,\Theta) - f(\cos\Theta) = \sum_{k=1}^{n} (f(x_k) - f(x))A_k(\Theta)$$
$$= \sum_{k=1}^{j-2} + U_{j-1} + U_j + U_{j+1} + \sum_{k=j+2}^{n} \quad . \tag{4.1}$$

where $\frac{(j-1)\pi}{n} < \Theta < \frac{j\pi}{n}$.

Of course if j=1 or 2 (or n-1,n) the first (or last) summation will not appear.

By (3.1) and (3.5),

$$|U_j| = |f(x_j) - f(x)|\ |A_j(\Theta)|$$
$$= O[\omega(\frac{\sin\Theta}{n}) + \omega(\frac{1}{n^2})] . \tag{4.2}$$

Similarly $|U_{j-1}|$ and $|U_{j+1}|$ can be estimated this way.

Let us now estimate

$$T \equiv \sum_{k=j+2}^{n} (f(x_k) - f(x)A_k(\Theta).$$

We shall employ a method of O. Kis [15] and group the summands in pairs.

If
$$B_k(\Theta) \equiv (f(x_k) - f(x))A_k(\Theta) + (f(x_{k+1}) - f(x))A_{k+1}(\Theta)$$
$$= (A_k(\Theta) + A_{k+1}(\Theta))(f(x_k) - f(x)) +$$
$$+ A_{k+1}(\Theta)(f(x_{k+1}) - f(x_k)) , \tag{4.3}$$

then

$$T = \sum_{i \in I} B_{j+i} + [A_n(\Theta)(f(x_n) - f(x))]$$

where $I = \{i: i = 2,4,6...; i < n - j\}$. The last term is written in brackers to signify that it appears only if

$(n-j)$ is even. For convenience, we shall drop the notation "$i \in I$" in the following calculations.

$$|\Sigma B_{j+i}(\Theta)| \le \Sigma |B_{j+i}(\Theta)| = \sum_{i\le j} |B_{j+i}(\Theta)| + \sum_{i>j} |B_{j+i}(\Theta)| .$$

Let us now consider $0 < \Theta \le \frac{\pi}{2}$. Then $\frac{c_1 j}{n} < \sin \Theta < \frac{c_2 j}{n}$. (4.4)

By appropriate parts of the four lemmas, the properties of ω, (4.3) and (4.4),

$$\sum_{i\le j} |B_{j+i}(\Theta)| = 0\Big(\sum_{i\le j} \frac{1}{i^3}[\omega(\frac{i \sin \Theta}{n}) + \omega(\frac{i^2}{n^2})]\Big) +$$

$$+ 0\Big(\sum_{i\le j} \frac{1}{i^2}[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{i}{n^2})]\Big)$$

$$= 0\Big(\sum_{i\le j} \frac{1}{i^2}[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{j}{n^2})]\Big)$$

(for $\omega(\frac{i}{n^2}) \le \omega(\frac{j}{n^2}) = 0(\omega(\frac{\sin \Theta}{n})$ from the definition of Θ.)

$$= 0(\omega(\frac{\sin \Theta}{n})) . \tag{4.5}$$

Now if $i > j$ then

$$\omega(\frac{i}{n^2}) \le \frac{2i}{j}\, \omega(\frac{j}{n^2}) .$$

Then if $0 < \Theta \le \frac{\pi}{2}$ by using the lemmas again

$$\sum_{i>j} |B_{j+i}(\Theta)| \le c \sum_{i>j} |A_k(\Theta) + A_{k+1}(\Theta)| (\omega(\frac{i \sin \Theta}{n}) + \omega(\frac{i^2}{n^2}))$$

$$+ c \sum_{i>j} |A_{k+1}(\Theta)| [\omega(\frac{\sin \Theta}{n}) + \omega(\frac{i}{n^2})]$$

$$= 0\Big[\sum_{i>j} \frac{1}{i^3}\, \omega(\frac{i \sin \Theta}{n}) + \sum_{i>j} \frac{n \sin \Theta}{i^4}\, \omega(\frac{i^2}{n^2})$$

$$+ \sum_{i>j} \frac{1}{i^2}\, \omega(\frac{\sin \Theta}{n}) + \sum_{i>j} \frac{n \sin \Theta}{i^3}\, \omega(\frac{i}{n^2})\Big]$$

$$= O[\omega(\frac{\sin\Theta}{n}) + \sum_{i>j} \frac{n \sin\Theta}{i^3} \omega(\frac{i}{n^2})]$$

$$= O[\omega(\frac{\sin\Theta}{n}) + \sum_{i>j} \frac{n \sin\Theta}{i^3} \frac{i}{j} \omega(\frac{j}{n^2})]$$

$$= O(\omega(\frac{\sin\Theta}{n})) \quad . \tag{4.6}$$

Here we make use of (4.4). Thus by (4.5) and (4.6) we have shown that if $0 < \Theta \le \frac{\pi}{2}$ then

$$\sum_{i\in I} |B_{j+i}(\Theta)| = O(\omega(\frac{\sin\Theta}{n}).$$

Then case $\frac{\pi}{2} \le \Theta < \pi$ is easier to consider. Then,

$$\frac{i\pi}{n} = \frac{(k-j)\pi}{n}$$

$$\le \frac{(n-j)\pi}{n} = O(\sin\Theta).$$

Hence

$$\sum_{i\in I} |B_{j+i}(\Theta)| = O[\sum_{i\in I} \frac{1}{i^3}(\omega(\frac{i\sin\Theta}{n}) + \omega(\frac{i^2}{n^2}))]$$

$$+ O[\sum_{i\in I} \frac{1}{i^2}(\omega(\frac{\sin\Theta}{n}) \cdot$$

Thus if $\frac{(j-1)\pi}{n} < \Theta < \frac{j\pi}{n}$ then

$$\sum_{i\in I} |B_{j+i}(\Theta)| = O(\omega(\frac{\sin\Theta}{n}))\cdot$$

If n-j is even then the term $|A_n(\Theta)||f(x_n) - f(x)|$ must be estimated. By (3.2) and (3.6),

$$|A_n(\Theta)||f(x_n) - f(x)| = O[\omega(\frac{\sin\Theta}{n}) +$$

$$+ \omega(\frac{1}{n^2})].$$

Therefore we have shown that

$$\sum_{k=j+2}^{n} (f(x_k) - f(x))\, A_k(\Theta) = O[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{1}{n^2})] \quad .$$

Similar calculations show that

$$\sum_{k=1}^{j-2} (f(x_k) - f(x)) A_k(\Theta) = O[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{1}{n^2})] \ .$$

Thus recalling (4.1) and (4.2),

$$|G_n(f,\Theta) - f(\cos \Theta)| = O[\omega(\frac{\sin \Theta}{n}) + \omega(\frac{1}{n^2})]$$

and the proof is complete.

CHAPTER V

A NEW PROOF OF A THEOREM OF S. A. TELJAKOVSKII

1. Introduction

There are several direct theorems in approximation theory. Here we are concerned with the following theorem of S. A. Teljakovskii [25] and I. E. Gopengauz [11] regarding approximation of functions defined on the segment $[-1,+1]$ by means of algebraic polynomials.

Theorem 1. Let a function f be continuous on $[-1,+1]$. Then for each $n = 1,2,3,\ldots$ there exists an algebraic polynomial P_n of degree n such that for all $x \in [-1,+1]$ the following estimate holds:

$$|f(x) - P_n(x)| \le R\,\omega\left(\frac{\sqrt{(1-x^2)}}{n}\right) \qquad (1.1)$$

where $\omega(\delta)$ is the modulus of continuity of f, and R is an absolute constant. Inequality (1.1) sharpens the well known Theorem of A. F. Timan [26] and was originally conjectured to S. A. Teljakovskii by S. B. Stechkin.

In this chapter we present a new proof of Theorem 1. In the second part of this chapter we attempt to answer the following question: Does there exist an algebraic polynomial of degree $\le$ n-1 depending on only n values of

$f(x)$ for which

$$f(x+h) - 2f(x) + f(x-h) = O(h) \qquad (1.2)$$

implies

$$|f(x) - P_n(x)| = O(\tfrac{1}{n})?$$

Problems of this kind were initially raised by A. Zygmund [3] and P. L. Butzer [4].

2. The Main Theorem

We shall first describe the construction of the polynomial $H_n[f,x]$. Let

$$x_{kn} = \cos\frac{(2k-1)\pi}{2n}, \quad k = 1,2,\ldots,n \qquad (2.1)$$

be the zeros of

$$T_n(x) = \cos n\Theta, \quad x = \cos\Theta, \quad n = 1,2,\ldots$$

the Chebyshev polynomial of the first kind. The fundamental polynomials of Lagrange interpolation constructed on the nodes (2.1) are given by

$$\ell_{kn}(\Theta) = \frac{(-1)^{k+1}\cos n\Theta \sin\Theta_{kn}}{n\,(\cos\Theta - \cos\Theta_{kn})} \quad k = 1,2,\ldots n. \qquad (2.2)$$

$\ell_{kn}(\Theta)$ is an algebraic polynomial in x of degree at most n-1. Further, as in Chapter 4, let

$$G_n[f] = G_n[f,x] = \sum_{k=1}^{n} f(x_{kn})A_{kn}(\Theta),$$

where, using the notation in (2.2),

$$2A_{kn}(\Theta) = \ell_{kn}(\Theta + \tfrac{\pi}{2n}) + \ell_{kn}(\Theta - \tfrac{\pi}{2n}). \qquad (2.3)$$

Set

$$H_n[f,x] = G_n[f,x] - (\tfrac{1+x}{2})(G_n[f,1] - f(1)) - \frac{(1-x)}{2}(G_n[f,-1] - f(-1)). \qquad (2.4)$$

Now we state the main theorem of this chapter.

Theorem 2. Let $f \in C[-1,+1]$ and we denote by $\omega(\delta)$, $\omega_2(\delta)$ its first and second modulus of continuity respectively. Then we have

$$|H_n[f,x] - f(x)| = 0(\omega(\frac{\sqrt{(1-x^2)}}{n}), \qquad (2.5)$$

and

$$|H_n[f,x] - f(x)| = 0(\omega_2(\frac{1}{n})). \qquad (2.6)$$

Corollary. Let $f(x)$ satisfy condition (1.2). Then we have $|H_n[f,x] - f(x)| = 0(\frac{1}{n})$.

This is an immediate consequence of (2.6), and (1.1).

3. Preliminaries

In Chapter 4 it was proved that

$$|G_n[f,x] - f(x)| = 0[w(\frac{\sqrt{(1-x^2)}}{n}) + w(\frac{1}{n^2})]. \quad (3.1)$$

For our purpose we need the following lemmas.

Lemma 1. If $0 < \Theta < \frac{\pi}{n}$ and $2 \le k \le n$ then we have

$$|A_{kn}(\Theta)| = 0(\frac{n \sin\Theta}{k^3}),$$

where $A_{kn}(\Theta)$ is defined by (2.3). For the proof of this lemma apply Lemma 3 in Chapter 4.

Lemma 2. If $0 < \Theta < \frac{\pi}{n}$ and $2 \le k \le n-1$ then we have

$$|A_{kn}(\Theta) + A_{k+1,n}(\Theta)| = 0(\frac{n \sin\Theta}{k^4}).$$

For the proof of this lemma apply Lemma 4 in Chapter 4.

Lemma 3. If $0 < \Theta < \frac{\pi}{n}$ and $2 \le k \le n$ then we have

$$|f(x_{kn}) - f(x_{1n})| = 0(\frac{k^2}{n \sin\Theta}\,\omega(\frac{\sin\Theta}{n})) .$$

Further, if $0 < \Theta < \frac{\pi}{n}$ and $2 \le k \le n-1$ then we have

$$|f(x_{kn}) - f(x_{k+1,n})| = 0(k\ \omega(\frac{\sin\Theta}{n})) .$$

For the proof of this lemma apply Lemmas 1 and 2 in Chapter 4 and use the properties of ω.

Finally, we need

Lemma 4. If $0 \le \Theta$ and $\sin\Theta < \frac{1}{n}$ then

$$S(\Theta) \equiv \sum_{k=2}^{n} (f(x_{kn}) - f(x_{1n}))A_{kn}(\Theta) = 0(\omega(\frac{\sin\Theta}{n})) .$$

Proof. For each k satisfying $2 \le k \le n-1$ let

$$B_{kn}(\Theta) \equiv (f(x_{kn}) - f(x_{1n}))A_{kn}(\Theta) + (f(x_{k+1,n})-f(x_{1n}))A_{k+1,n}(\Theta)$$

$$= (A_{kn}(\Theta) + A_{k+1,n}(\Theta))\ (f(x_{kn}) - f(x_{1n}) +$$

$$+ (f(x_{k+1,n}) - f(x_{kn}))A_{kn}(\Theta) .$$

Then

$$S(\Theta) = \sum_{i} B_{in}(\Theta) + [A_{nn}(\Theta)(f(x_{nn}) - f(x_{1n})] ,$$

where i ranges over the set of even natural numbers less than or equal to n-1. The last term is written in brackets to signify that it may or may not appear depending on the parity of n. Now, on using Lemma's 1, 2, and 3 the desired result follows.

4. Proof of Theorem 2.

The proof of (2.5) is divided into two natural parts. First, we consider the case when $\sqrt{(1-x^2)} \ge \frac{1}{n}$. Then, from (3.1), it follows that

$$|G_n[f,x] - f(x)| = O[\omega(\frac{\sqrt{(1-x^2)}}{n}) + \omega(\frac{1}{n^2})] ,$$
$$= O[\omega(\frac{\sqrt{(1-x^2)}}{n}) + \omega \frac{\sqrt{(1-x^2)}}{n}]$$
$$= O[\omega(\frac{\sqrt{(1-x^2)}}{n})] . \qquad (4.1)$$

on using (2.4) and (4.1) we obtain

$$|H_n[f,x] - f(x)| \le |G_n[f,x] - f(x)| +$$
$$+ |f(1) - G_n[f,1]|(\frac{1+x}{2}) + |f(-1)-G_n[f,-1]|\frac{(1-x)}{2}$$
$$= O\ (\omega(\frac{\sqrt{(1-x^2)}}{n}) . \qquad (4.2)$$

Secondly we consider the case when $0 < \sqrt{(1-x^2)} < \frac{1}{n}$. We may suppose that $x > 0$ since a similar proof can be constructed for $x < 0$. A simple computation shows that

$$H_n[f,x] - f(x) = \sum_{k=2}^{n} (f(x_{kn}) - f(x_{in}))A_{kn}(\theta) +$$
$$+ (f(x_{1n}) - f(x_{nn}))\ (\frac{1-x}{2}) + (f(-1) - f(1))\ \frac{1-x}{2} +$$
$$+ (f(1) - f(x)). \qquad (4.3)$$
$$= I_1 + I_2 + I_3 + I_4 \quad \text{(say)}.$$

For $x > 0$, $\sqrt{(1-x^2)} < \frac{1}{n}$ we have

$$\omega(1-x) \le \omega(1-x^2) \le \omega(\frac{\sqrt{(1-x^2)}}{n}) .$$

Therefore

$$|I_4| = |f(1) - f(x)| \le \omega(1-x) \le \omega(\frac{\sqrt{(1-x^2)}}{n}) . \qquad (4.4)$$

Furthermore,

$$|f(x_{1n}) - f(x_{nn})|\frac{1-x}{2} \le \frac{1-x}{2}\ \omega(x_{1n}-x_{nn}) \le (\frac{1-x}{2})\omega(2)$$
$$\le 2\omega(1-x) \le 2\omega(\frac{\sqrt{(1-x^2)}}{n}) .$$

This proves that

$$|I_4| = O[\omega(\frac{\sqrt{(1-x^2)}}{n})] . \qquad (4.5)$$

Similarly, we have

$$|I_3| = O[\omega(\frac{\sqrt{(1-x^2)}}{n})]. \tag{4.6}$$

Lastly, on using Lemma 4 we have

$$|I_1| = O[\omega(\frac{\sqrt{(1-x^2)}}{n})].$$

On combining (4.2) - (4.7) we have (2.5).

The proof of (2.6) is very similar to that of Theorem 1 in Chapter 4. Thus Theorem 2 has been proved.

It is of some interest to raise the following question: Is it true that $|H_n[f,x] - f(x)| = O(\omega_2(\frac{\sqrt{(1-x^2)}}{n}))$? Unfortunately, I have not been able to answer this question.

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BIOGRAPHICAL SKETCH

Terence M. Mills was born in Sydney, Australia, in 1948. He was later educated at the University of Sydney. He then taught and studied at the University of Melbourne from which he graduated as a Master of Arts with Honours. Since the Fall of 1970 he has been teaching and studying at the University of Florida. His private activities are of little interest to anyone except his wife and son.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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